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# Excitation spectrum and critical exponents of a one-dimensional integrable model of fermions with correlated hopping\*

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**Abstract.** We investigate the excitation spectrum of a model of  $N$  colour fermions with correlated hopping which can be solved by a nested Bethe ansatz. The gapless excitations of particle-hole type are calculated as well as the spin-wave like excitations which have a gap. Using general predictions of conformal field theory the long distance behaviour of some groundstate correlation functions are derived from a finite-size analysis of the gapless excitations. From the algebraic decay we show that for increasing particle density the correlation of so-called  $N$ -multiplets of particles dominates over the density-density correlation. This indicates the presence of bound complexes of these  $N$ -multiplets. This picture is also supported by the calculation of the effective mass of charge carriers.

## 1. Introduction

The study of low-dimensional electronic systems with strong correlations has gained considerable importance due to the discovery of high-temperature superconductivity. The models which have been mostly studied are the one-dimensional Hubbard model [1] and the  $t$ - $J$  model [2, 3] at its supersymmetric point. These models are special because of their integrability [4–7]. A great deal of current research is directed to more extensive rigorous and exact results on correlated systems in one and two dimensions. However, in the latter case exact results are sparse. In contrast to this we have a different situation in one dimension where many models are integrable and show different physical behaviour.

In this paper we continue the study of a fermionic model with correlated hopping [8]. In addition to on-site Coulomb interactions as in the above mentioned cases we are interested in modifications of the hopping terms of fermions in the vicinity of other particles. Unfortunately, such complex models are not integrable in general. However, for purely correlated hopping the situation looks much better. In [9] an integrable model of electrons was found which was generalized to fermions with arbitrary number  $N \geq 2$  of colours [10]. The Hamiltonian of this model is given by

$$H = - \sum_{j=1}^L \sum_{\tau=1}^N (c_{j,\tau}^{\dagger} c_{j+1,\tau} + c_{j+1,\tau}^{\dagger} c_{j,\tau}) \exp \left( -\eta \sum_{\substack{\tau'=1 \\ \tau' \neq \tau}}^N n_{j+\theta(\tau-\tau'),\tau'} \right) \quad (1)$$

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where  $L$  is the length of the chain,  $c_{j,\tau}^+$  and  $c_{j,\tau}$  are the creation and annihilation operators of fermions of colour  $\tau$  at site  $j$  ( $\tau = 1, \dots, N$  and  $j = 1, \dots, L$ ),  $n_{j,\tau}$  is the number operator and  $\theta$  is the step function

$$\theta(\tau - \tau') = \begin{cases} 1 & \tau > \tau' \\ 0 & \tau < \tau'. \end{cases} \quad (2)$$

We employ the usual periodic boundary conditions  $c_{L+1,\tau} = c_{1,\tau}$ . Repulsion of particles corresponds to an interaction parameter  $\eta > 0$ . In [8] the case  $N = 2$  of model (1) was investigated, however, for negative values of  $\eta$ . Note that  $\eta$  and  $-\eta$  are related by a particle-hole transformation. Using a Jordan-Wigner transformation the model can be formulated equivalently as a system of  $N$  interacting  $XY$  chains [10].

In section 2 we give a summary of the Bethe Ansatz equations and some groundstate properties. In section 3 we study the excitations of the model. In section 4 we derive the long-distance behaviour of correlation functions and compare the results with previous ones [8] for the special case of  $N = 2$ . In the appendix a short-cut to the analysis of Bethe Ansatz equations is presented on the basis of inversion identities.

## 2. The Bethe ansatz

The Bethe ansatz for this model has been derived in [10] from which we quote the relevant equations. The eigenstates of the Hamiltonian are characterized by sets of wavenumbers  $k_j$ ,  $j = 1, \dots, \mathcal{N}$ , for  $\mathcal{N}$  particles. There are additional Bethe ansatz parameters  $\Lambda_\alpha^{(r)}$  ( $r = 1, \dots, N - 1$  and  $\alpha = 1, \dots, M_{N-r}$ ) the number of which is given by

$$M_{N-r} = \sum_{j=1}^{N-r} \mathcal{N}_j \quad (3)$$

where  $\mathcal{N}_j$  denotes the number of particles of colour  $j$ . The parameters  $k_j$  and  $\Lambda_\alpha^{(r)}$  satisfy the set of nested Bethe ansatz equations derived in [10]

$$k_j L + \sum_{\alpha=1}^{M_{N-1}} \Theta\left(k_j - \Lambda_\alpha^{(1)}; \frac{\eta}{2}\right) = 2\pi I_j \quad (4)$$

$$\sum_{\sigma=\pm 1} \sum_{\beta=1}^{M_{N-r-\sigma}} \Theta\left(\Lambda_\alpha^{(r)} - \Lambda_\beta^{(r+\sigma)}; \frac{\eta}{2}\right) - \sum_{\beta=1}^{M_{N-r}} \Theta\left(\Lambda_\alpha^{(r)} - \Lambda_\beta^{(r)}; \eta\right) = 2\pi J_\alpha^{(N-r)}$$

with  $j = 1, \dots, \mathcal{N}$ ;  $r = 1, \dots, N - 1$ ;  $\alpha = 1, \dots, M_{N-r}$ , and we have set  $\Lambda_j^{(0)} = k_j$ . The phase shift function  $\Theta$  is given by

$$\Theta(k; \eta) = 2 \tan^{-1}\left(\coth \eta \tan \frac{1}{2}k\right) \quad -\pi \leq \Theta < \pi \quad (5)$$

and  $I_j$  and  $J_\alpha^{(r)}$  are integer (half-integer) numbers for odd (even)  $M_{N-1} + 1$  and  $M_{r-1} + \mathcal{N}_{r+1}$ , respectively. The energy and momentum of the corresponding state are given by

$$E = -2 \sum_{j=1}^{\mathcal{N}} \cos k_j \quad (6)$$

$$P = \sum_{j=1}^{\mathcal{N}} k_j = \frac{2\pi}{L} \left( \sum_{j=1}^{\mathcal{N}} I_j + \sum_{r=1}^{N-1} \sum_{\alpha=1}^{M_{N-r}} J_\alpha^{(N-r)} \right).$$

It follows by symmetry that the ground state corresponds to symmetric configurations with the same number of particles  $\mathcal{N}_i$  for all colours  $i$

$$\mathcal{N}_i = \frac{\mathcal{N}}{N} \quad 1 \leq i \leq N. \quad (7)$$

The ground state energy is calculated from (4)–(6) in the thermodynamic limit [10]

$$\frac{E_0}{L} = -2 \int_{-K}^K \cos k \rho(k) dk \quad (8)$$

where the density function  $\rho(k)$  is determined as the solution of the integral equation

$$2\pi\rho(k) - \int_{-K}^K \varphi(k-k')\rho(k') dk' = 1 \quad (9)$$

$$\varphi(k) = 1 - \frac{1}{N} + 2 \sum_{n=1}^{\infty} \exp(-n\eta) \frac{\sinh[n\eta(N-1)]}{\sinh(n\eta N)} \cos(nk)$$

and  $K$  is determined by the subsidiary condition

$$\int_{-K}^K \rho(k) dk = \frac{\mathcal{N}}{L} = \rho. \quad (10)$$

Equations (8)–(10) determine the ground state energy of the model as a function of the particle density  $\rho$ . The appendix contains a derivation of (8)–(10) from (4)–(6) which is slightly different from the original treatment in [10].

In (7) we have assumed  $\mathcal{N}$  a multiple of  $N$ . This in fact is the condition for a proper accommodation of the ‘antiferromagnetic’ ground state. For particle numbers  $\mathcal{N}$  not multiples of  $N$  we always have a misfit resulting in ‘spin-type’ excitations which are treated below. This kind of excitations has a non-zero gap by which the bulk energies of the ground states of systems with  $\mathcal{N}$  not multiples of  $N$  are increased.

### 3. Excited states

The system of Bethe ansatz equations (4) admits many solutions depending on the choice of the parameters  $I_j$  and  $J_\alpha^{(r)}$ . In particular the solution for the ground state (8) is characterized by a parameter set in which the  $I_j$  and  $J_\alpha^{(r)}$  are consecutive integers (or half-integers) centred around the origin. It is expected that the low-lying states are obtained by small modifications to these configurations.

There are two types of elementary excitations leaving the particle number unchanged. The first one is of particle-hole type corresponding to raising one pseudo-particle in the ground state  $|k_h| < K$  to a higher level  $|k_p| > K$ . This excitation is obtained by a straight forward manipulation of the set of numbers  $I_j$  without changing the parameters  $J_\alpha^{(r)}$ . This produces gapless excitations analogous to those in the antiferromagnetic spin- $\frac{1}{2}$  Heisenberg chain [11, 12] and in the repulsive Hubbard model [13–15]. In the thermodynamic limit energy and momentum of these excitations are given by

$$E - E_0 = \sum_{j=1}^{\nu} \epsilon(k_j^p) - \sum_{j=1}^{\nu} \epsilon(k_j^h) \quad (11)$$

$$P - P_0 = \sum_{j=1}^{\nu} p(k_j^p) - \sum_{j=1}^{\nu} p(k_j^h)$$

with energy-momentum dispersion of particle-hole excitations

$$\begin{aligned}\epsilon(\vartheta) &= -2 \left[ \cos \vartheta + \int_{-K}^K \sin k j(k) dk \right] \\ p(\vartheta) &= \vartheta - \int_{-K}^K j(k) dk.\end{aligned}\tag{12}$$

The function  $j(k)$  is the solution of the integral equation

$$2\pi j(k) - \int_{-K}^K \varphi(k-k') j(k') dk' = \Phi(k-\vartheta)\tag{13}$$

where  $\Phi'(k) = \varphi(k)$ . Alternatively, the dressed energy function  $\epsilon$  can be obtained directly from an integral equation identical to the last one upon the replacement of  $j(k)$  and  $\Phi(k-\vartheta)/2\pi$  by  $\epsilon(k)$  and  $\epsilon_0(k) = -2 \cos k$ , respectively. In general (13) can be solved only numerically. In certain limiting cases also analytic results can be obtained. For instance, in the strong-coupling limit ( $\eta \rightarrow \infty$ ) perturbation theory yields

$$\begin{aligned}\epsilon(\vartheta) &= -2 \left[ 1 + \frac{e^{-2\eta}}{\pi} \left( K - \frac{1}{2} \sin 2K \right) \right] \left( \cos \vartheta + \frac{t}{\pi} (\sin K - K \cos K) \right) + O(e^{-2\eta}) \\ p(\vartheta) &= \frac{\pi}{\pi - Kt} \vartheta + \frac{2e^{-2\eta} \sin K}{\pi - Kt} \left[ \sin \vartheta + \frac{t \sin K}{\pi - Kt} \vartheta \right] + O(e^{-4\eta}) \quad t = \frac{N-1}{N}.\end{aligned}\tag{14}$$

The excitations of the second type involve 'r-holes' and 'strings' in the sets  $\Lambda_\alpha^{(r)}$ . A 'r-hole' occurs when there is a jump in the sequence of  $J_\alpha^{(r)}$ . Physically it describes states where the colour of one bare particle has been changed. In a sense these excitations are of spin-wave type, but have a non-zero gap. The general energy-momentum excitation is given by  $N-1$  different dispersion functions

$$\begin{aligned}E - E_0 &= \sum_{r=1}^{N-1} \sum_{j=1}^{v_{N-r}} \epsilon^{(N-r)}(\vartheta_j^{(r)}) \\ P - P_0 &= \sum_{r=1}^{N-1} \sum_{j=1}^{v_{N-r}} p^{(N-r)}(\vartheta_j^{(r)}).\end{aligned}\tag{15}$$

Each elementary excitation parametrized with a rapidity variable  $\vartheta$  is given by

$$\begin{aligned}\epsilon^{(N-r)}(\vartheta) &= 2 \int_{-K}^K \sin k j(k) dk \\ p^{(N-r)}(\vartheta) &= \int_{-K}^K j(k) dk\end{aligned}\tag{16}$$

where  $j(k)$  is the solution of an integral equation of the type (13), however, with a different right-hand side

$$2\pi j(k) - \int_{-K}^K \varphi(k-k') j(k') dk' = \varphi^{(N-r)}(k-\vartheta)\tag{17}$$

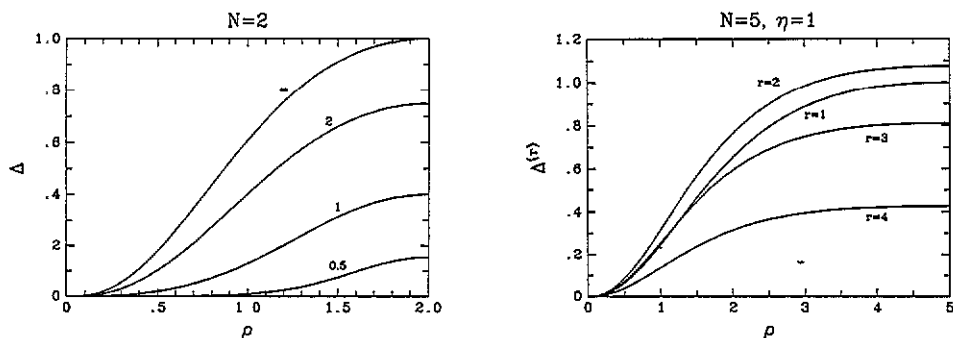


Figure 1. Left part: plot of the gap of the massive excitations as a function of the particle density  $\rho$  for  $N = 2$  and different interaction strengths  $\eta = 0.5, 1, 2, \infty$ . Right part: the excitation gap for the four massive dispersions in the case  $N = 5$  and  $\eta = 1$ .

where

$$\varphi^{(N-r)}(k) = \frac{N-r}{N}k + 2 \sum_{n=1}^{\infty} \frac{\sinh[n\eta(N-r)] \sin(nk)}{\sinh(n\eta N) n} \quad (18)$$

The lowest energies  $\Delta^{(r)}$  of (16) are taken at  $\vartheta = \pi$ . In figure 1 the gaps are shown for different values of  $N$  and  $\eta$  in dependence of the particle density  $\rho$ . Not all combinations of the elementary excitations (16) are allowed, the numbers  $\nu_r$  in (15) have to satisfy a selection rule

$$\sum_{r=1}^{N-1} r \nu_{N-r} \equiv 0 \pmod{N} \quad (19)$$

For a derivation of (17) and (18) utilizing inversion identities see the appendix. In general (16), (17) have to be solved numerically. In the strong-coupling limit ( $\eta \rightarrow \infty$ ) the analytic solution of (16)–(18) is

$$\begin{aligned} \epsilon^{(N-r)}(\vartheta) &= \Delta^{(N-r)} + \frac{2e^{-r\eta}}{\pi} \left( K - \frac{1}{2} \sin 2K \right) (1 + \cos \vartheta) + O(e^{-(r+2)\eta}) \\ p^{(N-r)}(\vartheta) &= \frac{K}{\pi - Kt} \frac{N-r}{N} (\pi - \vartheta) - \frac{2T_1 \sin K}{\pi - Kt} + O(e^{-4\eta}) \\ T_1 &= e^{-r\eta} \sin \vartheta - e^{-2\eta} \frac{N-r}{N} \frac{\pi - \vartheta}{\pi - Kt} \sin K \\ \Delta^{(N-r)} &= 2T_2 \left[ 1 + \frac{e^{-2\eta}}{\pi} \left( K - \frac{1}{2} \sin 2K \right) (1 - T_2^{-1} e^{(2-r)\eta}) \right] \\ T_2 &= \frac{N-r}{\pi N} (\sin K - K \cos K). \end{aligned} \quad (20)$$

In addition to simple holes excited states may also contain conjugate pairs of complex  $\Lambda_{\alpha}^{(r)}$ . The additional ‘strings’ do not contribute to energy and momentum which effect is very transparent in the inversion identity approach, see the appendix. Such a cancellation

phenomenon is well known for the Heisenberg model, see, for instance, [12]. Concerning the spectrum of the model these states only lead to a degeneracy of the energy levels. With respect to the classification of all eigenstates, however, they do play an important role.

Thus in the considered model there are no gapless excitations of spin wave type in contrast to the antiferromagnetic Heisenberg chain [11–12] and the repulsive Hubbard model [13–15]. Instead the analogous excitations correspond to collective motions of pseudo-particles which can be regarded as new particles with non-zero masses.

The gap implies the binding of particles in multiplets of  $N$  particles with different colours. Any spatial separation of the constituents of these complexes would lead to two subsystems of the chain with particle numbers not multiples of  $N$ . As explained in section 2 this results in an increase of the ground state energies by the order of the ‘spin-gap’, and eventually leads to the exponential decay of the corresponding correlation function.

#### 4. Critical exponents of the correlation functions

In the Bethe ansatz approach it is a formidable task to deal with correlation functions. However, due to developments in two-dimensional conformal field theory the scaling dimensions describing the algebraic decay of correlation functions became accessible [16, 17]. According to this theory there is a one-to-one correspondence between the scaling dimensions and the spectrum of the quantum system on a (periodically closed) finite chain [18, 19]

$$E_j - E_0 = \frac{2\pi v_F}{L} (x_j + N^+ + N^-) \quad (21)$$

$$P_j - P_0 = \frac{2\pi}{L} (s_j + N^+ - N^-) + 2dk_F.$$

$E_j$  and  $P_j$  are the energy and momentum of the  $j$ th excited state,  $v_F$  is the Fermi velocity. The  $x_j$  and  $s_j$  are the scaling dimensions and ‘spins’ of the corresponding operators,  $N^+$ ,  $N^-$  are non-negative integers and  $d$  is the number of particles excited from the left Fermi point to the right one. The ground state energy is expected to scale as [20, 21]

$$E_0 = L\epsilon_\infty - \frac{\pi v_F}{6L} c \quad (22)$$

where  $\epsilon_\infty$  is the ground state energy per site of the infinite system and  $c$  is the central charge characterizing the underlying conformal field theory.

Consequently the scaling dimensions are obtained from the gaps due to finite-size effects in the spectrum of the Hamiltonian at criticality. In order to compute the finite-size corrections we use a method which has been developed by previous authors for the Heisenberg and Hubbard chains [22, 23]. As a result we have obtained the ground state energy (22) with central charge  $c = 1$  and the finite-size spectrum of the low-lying excitations

$$E - E_0 = \frac{2\pi v_F}{L} \left( \left[ \frac{\Delta\mathcal{N}}{2\xi(K)} \right]^2 + [\xi(K)]^2 d^2 + N^+ + N^- \right) \quad (23)$$

$$P = 2k_F d + \frac{2\pi}{L} (d\Delta\mathcal{N} + N^+ - N^-) \quad k_F = \pi\rho.$$

Here the dressed charge at the Fermi surface  $\xi(K)$  is related in a simple way to the function  $\rho(k)$  in (9)

$$\xi(K) = 2\pi\rho(K) \tag{24}$$

and  $\Delta\mathcal{N}$  is the change of the particle number compared to the ground state. The non-negative integers  $N^\pm$  describe the excitations of particle-hole type in the vicinity of the Fermi points  $\pm k_F$ . Now we can read off all critical exponents from (23) and (21)

$$x_{\Delta\mathcal{N},d} = \frac{(\Delta\mathcal{N})^2}{[2\xi(K)]^2} + [\xi(K)]^2 d^2 + N^+ + N^- \quad s_{\Delta\mathcal{N},d} = d\Delta\mathcal{N} + N^+ - N^- \tag{25}$$

where  $\Delta\mathcal{N}$  and  $d$  can take the values

$$\Delta\mathcal{N} = 0, N, 2N, \dots \quad d \equiv \frac{\Delta\mathcal{N}}{2N} \pmod{\frac{1}{N}} \tag{26}$$

generalizing the result in [24] for  $N = 2$ . The omitted integer values of  $\Delta\mathcal{N}$  correspond to excitations with gap which are not related to correlations with algebraic decay. The two-point correlation functions of the scaling fields  $\Phi_{\Delta^\pm}(x, t)$  with conformal weights  $\Delta^\pm = (x \pm s)/2$  are known to be [16, 19]

$$\langle \Phi_{\Delta^\pm}(x, t) \Phi_{\Delta^\pm}(0, 0) \rangle = \frac{\exp(-2idk_F x)}{(x - v_F t)^{2\Delta^+} (x + v_F t)^{2\Delta^-}} \tag{27}$$

The dimensions of the descendant fields differ from  $\Delta^\pm$  by the integers  $N^\pm$ .

Let us first consider the density-density correlation function. In this case the correlation function is determined by excitations with unchanged number of particles  $\Delta\mathcal{N} = 0$ . The leading contributions to the algebraic decay of the correlation are then given by  $d = 0$ ,  $(N^+, N^-) = (1, 0)$  or  $(0, 1)$ , and by  $d = 1/N$ ,  $N^\pm = 0$ , respectively. This leads to the asymptotic form

$$\langle \rho(r) \rho(0) \rangle \simeq \rho_0^2 + A_1 r^{-2} + A_2 r^{-\alpha} \cos((2/N)k_F r) \tag{28}$$

where

$$\rho(r) = \sum_{\tau=1}^N n_{r,\tau} \quad \alpha = \frac{2\xi(K)^2}{N^2} \tag{29}$$

Field-field correlators decay exponentially as the corresponding excitations change the particle number  $\mathcal{N}$  by one unit, as remarked before. However, we may consider the correlation function of a  $N$ -multiplet of fields,

$$O(r) = \prod_{\tau=1}^N c_{r,\tau} \tag{30}$$

In this case the particle number changes by  $\Delta\mathcal{N} = N$  which according to (26) is the smallest non-zero value for which we have algebraic decay. We have  $d = 0$  or  $1/2N$  for even or odd  $N$ , respectively. We thus obtain

$$\langle O^+(r) O(0) \rangle \simeq r^{-\beta} \begin{cases} 1 & N \text{ even} \\ \cos((k_F/N)r) & N \text{ odd} \end{cases} \quad \beta = \begin{cases} 1/\alpha & N \text{ even} \\ 1/\alpha + \alpha/4 & N \text{ odd} \end{cases} \tag{31}$$



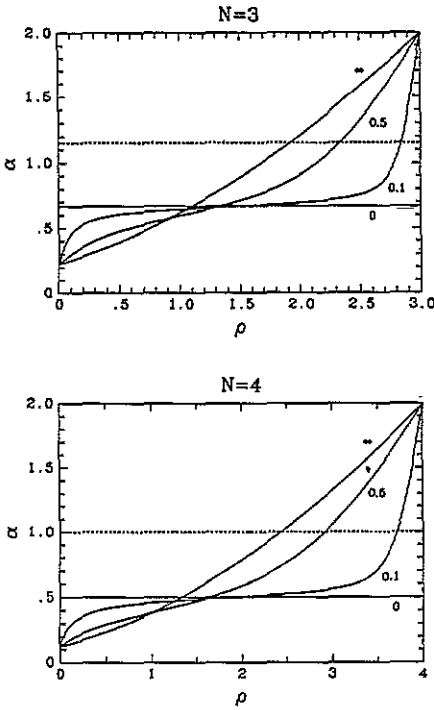


Figure 2. Upper part: plot of the critical exponent  $\alpha$  for  $N = 3$  and different values of  $\eta = 0, 0.1, 0.5, \infty$ . For values of  $\alpha$  above the broken line we have dominating  $N$ -multiplet correlations. Lower part: the same for  $N = 4$ .

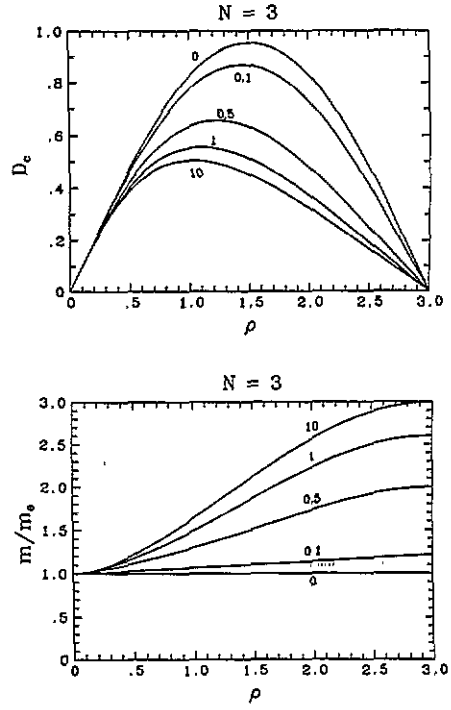


Figure 3. The charge stiffness (upper part), and the effective mass of the charge carriers (lower part) for  $N = 3$  and different interaction strengths  $\eta = 0, 0.1, 0.5, 1, 10$ . For particle densities close to 3 and large interactions  $\eta$  the mass approaches 3 times the bare mass  $m_e$ .

The characteristic exponent (29) is determined numerically and plotted in figure 2 for the cases  $N = 3, 4$ . For  $N = 2$  we refer to [8]. Analytically we find  $\alpha(\rho = 0) = 2/N^2$  and  $\alpha(\rho = N) = 2$ . This shows that for all  $N$  there exists an interval  $[\rho_c, N]$  for which  $\beta < \alpha$ . In this case the  $N$ -multiplet correlation (31) has a slower decay than the density correlation (28) and thus dominates. This might indicate the presence of bound complexes of  $N$ -multiplets in generalization of bound pairs for  $N = 2$  which has been discussed in [8].

In order to get a more complete picture of the physical properties of the model we also investigate the conductivity and the effective transport mass. Following the ideas of [25, 26] we study the Hamiltonian (1) with twisted boundary conditions with twisting angle  $\varphi$ . In this case the Bethe ansatz equations read

$$k_j L + \sum_{\alpha=1}^{M_{N-1}} \Theta\left(k_j - \Lambda_{\alpha}^{(1)}; \frac{\eta}{2}\right) = 2\pi I_j + \varphi$$

$$\sum_{\sigma=\pm 1} \sum_{\beta=1}^{M_{N-r-\sigma}} \Theta\left(\Lambda_{\alpha}^{(r)} - \Lambda_{\beta}^{(r+\sigma)}; \frac{\eta}{2}\right) - \sum_{\beta=1}^{M_{N-r}} \Theta\left(\Lambda_{\alpha}^{(r)} - \Lambda_{\beta}^{(r)}; \eta\right) = 2\pi J_{\alpha}^{(N-r)}.$$
(32)

The additional phase  $\varphi$  corresponds to an enclosed magnetic flux in the ring.

The conductivity is directly proportional to the charge stiffness (Drude weight)  $D_c$  which can be obtained from the change

$$\Delta E_0 = D_c \varphi^2 / L \quad (33)$$

in the ground state energy for small  $\varphi$  [25]. Using the charge stiffness we can define an effective transport mass  $m$  by the relation

$$\frac{m}{m_e} = \frac{D_c^0}{D_c} \quad (34)$$

where  $D_c^0 = (N/\pi) \sin(\pi\rho/N)$  is the charge stiffness of the non-interacting ( $\eta = 0$ ) system and  $m_e$  is the (bare) electron mass.

Using the result (23) for the finite-size corrections with  $\Delta\mathcal{N} = 0$  and  $d = \varphi/2\pi$  we find

$$D_c = \frac{1}{2\pi} v_F \xi^2(K) = \frac{1}{2\pi} \epsilon'(K) \xi(K). \quad (35)$$

Using the integral equation (9) and (24) we can calculate  $D_c$  and  $m$  numerically for all densities  $\rho$ , see figure 3. For large particle densities  $\rho$  and strong interaction  $\eta$  we observe an increase of the mass  $m$  by a factor  $N$  as compared to the non-interacting case. This is consistent with and supports our above picture that bound complexes of  $N$ -multiplets are present in the system.

## Appendix

Here we describe the reduction of the Bethe ansatz problem (4) involving  $k_j$  and  $\Lambda_\alpha^{(r)}$  parameters to equations involving only the  $k_j$  variables. To this end we cite a set of equations which are equivalent to (4) and which appeared at an intermediate stage of the analysis in [10]

$$e^{ik_j L} = T^1 \left( \frac{i}{2} k_j + \frac{\eta}{2} \right). \quad (A1)$$

$T^1(u)$  is the first of a set of functions  $T^1(u), T^2(u), \dots, T^{N-1}(u)$ , satisfying the recursion

$$T^r(u) = \frac{1}{q^r(u)} \left[ q^r(u - \eta) + q^r(u + \eta) \frac{q^{r-1}(u - \eta/2)}{q^{r-1}(u + \eta/2)} T^{r+1}(u + \eta/2) \right] \quad (A2)$$

with  $T^N(u) \equiv 1$ . The  $q$ -functions are given by

$$q^r(u) = \prod_{\alpha=1}^{M_{N-r}} \sinh \left( u - \frac{i}{2} \Lambda_\alpha^{(r)} \right) \quad (A3)$$

in terms of the parameters  $\Lambda_\alpha^{(r)}$  which are subject to the condition that all  $T^r(u)$  be analytic functions with simple poles at  $(i/2)\Lambda_\alpha^{(r-1)} - \eta/2$ . This condition is equivalent to the set of Bethe ansatz equations (4). Note that  $\Lambda_0^{(r)} = k_j$  and  $M_N = \mathcal{N}$ .

On the right-hand side of (A2) one of the two summands usually dominates over the other. This provides the possibility to derive a simple functional equation for  $T^1(u)$  in product form

$$T^1(u)T^1(u + \eta) \cdots T^1(u + (N - 1)\eta) \simeq \frac{q^0(u - \eta/2)}{q^0(u + (N - 3/2)\eta/2)} \tag{A4}$$

where the right-hand side depends only on the  $k_j$  parameters and the corrections to this relation are exponentially small in the thermodynamic limit. For the ground state (A4) can be solved uniquely (apart from  $N$ th roots of unity) under the condition of analyticity, absence of zeros in the physical regime  $-(N - 1)\eta < \text{Re}(u) < \eta$ , and the pole structure at  $(i/2)k_j - \eta/2$ . The result can be written in the form

$$\ln T^1\left(\frac{i}{2}k + \frac{\eta}{2}\right) = i\frac{2\pi}{N}j + \frac{1}{i} \sum_{k=1}^N \Phi(k - k_j) \quad j = 0, 1, \dots, N - 1 \tag{A5}$$

where

$$\Phi(k) = \left(1 - \frac{1}{N}\right)(k - \pi) + 2 \sum_{n=1}^{\infty} \frac{\exp(-n\eta)}{n} \frac{\sinh[n\eta(N - 1)]}{\sinh(n\eta N)} \sin(nk). \tag{A6}$$

Using standard reasoning from this and (A1) one derives the integral equation (9). Note that  $\varphi(k) = \Phi'(k)$ . The absolute ground state of the system corresponds to the choice of  $j = 0$  in (A5). Other choices of  $j$  lead to excitations with energies of order  $1/L$  and give rise to a quantization of the parameter  $d$  in (26) in units of  $1/N$ .

For excitations of spin-wave type it is useful to define the excitation function

$$t(u) = \lim_{L \rightarrow \infty} \frac{T^1(u)}{T_{\max}^1(u)} \tag{A7}$$

for which (A4) directly implies

$$t(u)t(u + \eta) \cdots t(u + (N - 1)\eta) = 1. \tag{A8}$$

A first consequence of this functional equation is the double periodicity of the analytic continuation of  $t(u)$

$$t(u) = t(u + i\pi) \tag{A9}$$

$$t(u) = [t(u + \eta)t(u + 2\eta) \cdots t(u + (N - 1)\eta)]^{-1} = t(u + N\eta).$$

The elementary spin-wave excitations are characterized by zeros and poles which have to appear in pairs  $u_z$  and  $u_p$ . The distance of  $u_z$  and  $u_p$  has to be a multiple of  $\eta$  in order to lead to the cancellation (A8). We write

$$u_{z,p} = \frac{1}{2}(\eta + i\vartheta \mp r\eta/2) \quad r = 1, \dots, N - 1 \tag{A10}$$

where  $\vartheta$  is a rapidity variable and takes real numbers. The solution of (A8) under the condition of analyticity and the zero/pole structure of (A10) is

$$\ln t((i/2)k + \eta/2) = i\varphi^{(N-r)}(k - \vartheta) \tag{A11}$$

where the function on the right-hand side was defined already in (18). Again by standard reasoning and (A1) one finds the integral equation (17). It is worth noting that in this approach the distribution of  $\Lambda$ -parameters is quite irrelevant. Different patterns, in particular simple holes with or without strings, lead to the same function (A11).

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